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On (n,m)–groups for $n \geq 3m$

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Abstract. In this article two characterization of $(n,m)-$groups for $n \geq 3m$ are proved (The case $m = 1$ is proved in [4]).

1. Preliminaries

Definition 1.1 ([1])**.** Let $n \geq 2$ and let $(Q; A)$ be an n–groupoid. We say that $(Q; A)$ is a Dörnte n–group [briefly: n–group] iff is an n–semigroup and n–quasigroup as well (See also [9]).

Definition 1.2 ([2])**.** Let $n \geq m+1$ and $(Q; A)$ be an (n,m)–groupoid ($A : Q^n \to Q^m$). We say that $(Q; A)$ is an (n,m)–group iff the following statements hold:

(i) For every $i, j \in \{1, \ldots, n-m+1\}, i < j$, the following law holds
$$A(x_1^{i-1}, A(x_i^{i+n-1}), x_{i+n}^{2n-m}) = A(x_1^{j-1}, A(x_j^{j+n-1}), x_{j+n}^{2n-m})$$
[:$< i, j >$–associative law];

(ii) For every $i \in \{1, \ldots, n-m+1\}$ and for every $a_1^n \in Q$ there is exactly one $x_1^m \in Q^m$ such that the following equality holds
$$A(a_1^{i-1}, x_1^m, a_i^{n-m}) = a_{n-m+1}^n.$$

For $m = 1$ $(Q; A)$ is an n–group. Cf. [9].

Proposition 1.1 ([3])**.** *Let $(Q; A)$ be an (n,m)–groupoid and let $n \geq m+2$. Also let the following statements hold*

(1) *$(Q; A)$ is a (n,m)–semigroup (cf. (i) in definition 1.2);*

(2) *For every $a_1^n \in Q$ there is exactly one $x_1^m \in Q^m$ such that the following equality holds*
$$A(a_1^{n-m}, x_1^m) = a_{n-m+1}^n;$$

(3) *For every $a_1^n \in Q$ there is exactly one $y_1^m \in Q^m$ such that the following equality holds*
$$A(y_1^m, a_1^{n-m}) = a_{n-m+1}^n.$$

Then $(Q; A)$ is an (n,m)–group. See, also [10].

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Definition 1.3 ([6])**.** Let $n \geq 2m$ and let $(Q;A)$ be an (n,m)–groupoid. Let also $\mathbf{e}$ be a mapping of the set Q^{n-2m} into the set Q^m. Then, we say that $\mathbf{e}$ is an $\{1, n-m+1\}$–**neutral operation** of the (n,m)–groupoid $(Q;A)$ iff for every sequence a_1^{n-2m} over Q and for every $x_1^m \in Q^m$ the following equalities hold:

$$A(x_1^m, a_1^{n-2m}, \mathbf{e}(a_1^{n-2m})) = x_1^m$$

and

$$A(\mathbf{e}(a_1^{n-2m}), a_1^{n-2m}, x_1^m) = x_1^m.$$

For $m = 1$ $\mathbf{e}$ is an $\{1, n\}$–neutral operation of the n–groupoid $(Q;A)$. Cf. Chapter II in [9].

2. Auxiliary propositions

Proposition 2.1 ([7])**.** *Let $n \geq 2m$ and let (Q,A) be an (n,m)–groupoid. Further on, let the following statements hold:*

- (a) *The $< 1, n-m+1 >$ –associative law holds in $(Q;A)$;*
- (b) *For every $a_1^n \in Q$, there is **at least one** $x_1^m \in Q^m$ such that the equality $A(a_1^{n-m}, x_1^m) = a_{n-m+1}^n$ holds;*
- (c) *For every $a_1^n \in Q$, there is **at least one** $y_1^m \in Q^m$ such that the equality $A(y_1^m, a_1^{n-m}) = a_{n-m+1}^n$ holds.*

Then $(Q;A)$ has a $\{1, n-m+1\}$–neutral operation.

For $m = 1$: Prop.2.5-II in [9].

In this paper, among others, the following $< i, j >$–associative laws have the prominence:

$$A(A(x_1^n), x_{n+1}^{2n-m}) = A(x_1, A(x_2^{n+1}), x_{n+2}^{2n-m}) \tag{1L}$$

and

$$A(x_1^{n-m-1}, A(x_{n-m}^{2n-m-1}), x_{2n-m}) = A(x_1^{n-m}, A(x_{n-m+1}^{2n-m})). \tag{1R}$$

Proposition 2.2 ([5])**.** *Let $(Q;A)$ be an (n,m)–group,* $\mathbf{e}$ *its $\{1, n-m+1\}$–neutral operation (cf. 1.4), and let $n > 2m$. Then, for every $a_1^{n-2m}, b_1^{n-2m}, x_1^m \in Q$ and for all $i \in \{1, \ldots, n-2m+1\}$ the following equalities hold:*

$$A(x_1^m, b_i^{n-2m}, \mathbf{e}(a_1^{n-2m}), b_1^{i-1}) = A(\mathbf{e}(a_1^{n-2m}), a_1^{n-2m}, x_1^m) \tag{1}$$

and

$$A(b_i^{n-2m}, \mathbf{e}(b_1^{n-2m}), b_1^{i-1}, x_1^m) = A(x_1^m, a_1^{n-2m}, \mathbf{e}(a_1^{n-2m})). \tag{2}$$

Cf. Prop. 1.1-IV in [9].

Proposition 2.3 ([7])**.** *Let $n > m+1$ and let $(Q;A)$ be an (n,m)–groupoid. Also let*

- (α) *The (1L) [(1R)] law holds in $(Q;A)$;*

(β) *For every* $x_1^m, y_1^m, a_1^{n-m} \in Q$ *the following implication holds*

$$A(x_1^m, a_1^{n-m}) = A(y_1^m, a_1^{n-m}) \Rightarrow x_1^m = y_1^m$$

$$[A(a_1^{n-m}, x_1^m) = A(a_1^{n-m}, y_1^m) \Rightarrow x_1^m = y_1^m].$$

Then, $(Q;A)$ *is an* (n,m)*–semigroup.*

Proposition 2.4 ([8])**.** *Let* $n \geq 3m$ *and let* $(Q;A)$ *be an* (n,m)*–groupoid. Then the following statements are equivalent*

(i) $(Q;A)$ *is an* (n,m)*–group;*

(ii) *There is at least one* $i \in \{m+1, \ldots, n-2m+1\}$ *such that the following conditions hold:*

  (a) *the* $< i-1, i >$ *–associative law holds in* $(Q;A)$*;*

  (b) *the* $< i, i+1 >$ *–associative law holds in* $(Q;A)$*;*

  (c) *for every* $a_1^n \in Q$ *there is exactly one* $x_1^m \in Q^m$ *such that the following equality holds*

$$A(a_1^{i-1}, x_1^m, a_i^{n-m}) = a_{n-m+1}^n.$$

For $m = 1$*: Th. 3.4-IX in* [9]*.*

3. Results

Theorem 3.1. *Let* $n \geq 3m$ *and let* $(Q;A)$ *be an* (n,m)*–groupoid. Then,* $(Q;A)$ *is an* (n,m)*–group iff there is a mapping* E *of the set* Q^{n-2m} *into the set* Q^m *such that the laws*

(1L) $$A(A(x_1^n), x_{n+1}^{2n-m}) = A(x_1, A(x_2^{n+1}), x_{n+2}^{2n-m})$$

(1Lm) $$A(A(a_1^m, b_1^{n-m}), c_1^m, d_1^{n-2m}) = A(a_1^m, A(b_1^{n-m}, c_1^m), d_1^{n-2m}),$$

($\widehat{2\mathrm{L}}$) $$A(a_1^{n-2m}, \mathsf{E}(a_1^{n-2m}), x_1^m) = x_1^m$$

and

(2R) $$A(x_1^m, a_1^{n-2m}, \mathsf{E}(a_1^{n-2m})) = x_1^m$$

hold in the algebra $(Q;A,\mathsf{E})$*.*

Remark 3.1. For $m = 1$: (1L)=(1Lm).

Proof.

a) $\Rightarrow$ Let $(Q;A)$ be an (n,m)–group. Then, by Prop. 2.1, by Def. 1.2 and by Prop. 2.2, there is an algebra $(Q;A,\mathbf{e})$ of the type $< (n,m), (n-2m,m) >$ in which the laws (1L),(1Lm),($\widehat{2L}$) and (2R) hold.

b) $\Leftarrow$ Let $(Q;A,\mathbf{e})$ be an algebra of the type $< (n,m), (n-2m,m) >$ in which the laws (1L),(1Lm), ($\widehat{2L}$) and (2R) are satisfied. Firstly, we prove that under the assumptions the following statements hold:

1° For every $x_1^m, y_1^m, b_1^m \in Q^m$ and for every sequence a_1^{n-2m} over Q the following implication holds

$$A(x_1^m, b_1^m, a_1^{n-2m}) = A(y_1^m, b_1^m, a_1^{n-2m}) \Rightarrow x_1^m = y_1^m;$$

2° $(Q; A)$ is an (n,m)–semigroup;
3° $(\forall b_1^m \in Q^m)(\forall c_i \in Q)_1^{n-3m}\ b_1^m = \mathsf{E}(c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m}))$;
4° For every $x_1^m, y_1^m, b_1^m \in Q^m$ and for every sequence a_1^{n-2m} over Q the implication

$$A(b_1^m, x_1^m, a_1^{n-2m}) = A(b_1^m, y_1^m, a_1^{n-2m}) \Rightarrow x_1^m = y_1^m$$

holds;
5° For every $x_1^m, y_1^m, b_1^m \in Q^m$ and for every sequence a_1^{n-2m} over Q the implication

$$A(a_1^{n-2m}, x_1^m, b_1^m) = A(a_1^{n-2m}, y_1^m, b_1^m) \Rightarrow x_1^m = y_1^m$$

holds;
6° For every $x_1^m, b_1^m, c_1^m \in Q^m$ and for every sequence a_1^{n-2m} over Q the equivalence

$$\begin{aligned}&A(b_1^m, x_1^m, a_1^{n-2m}) = c_1^m \Leftrightarrow\\ &x_1^m = A(d_1^{n-3m}, \mathsf{E}(b_1^m, d_1^{n-3m}), c_1^m, \mathsf{E}(a_1^{n-2m})).\end{aligned}$$

Sketch of the proof of 1°:

$$\begin{aligned}&A(x_1^m, b_1^m, a_1^{n-2m}) = A(y_1^m, b_1^m, a_1^{n-2m}) \Rightarrow\\ &A(A(x_1^m, b_1^m, a_1^{n-2m}), \mathsf{E}(a_1^{n-2m}), c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m})) =\\ &A(A(y_1^m, b_1^m, a_1^{n-2m}), \mathsf{E}(a_1^{n-2m}), c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m})) \overset{(1\mathrm{Lm})}{\Longrightarrow}\\ &A(x_1^m, A(b_1^m, a_1^{n-2m}, \mathsf{E}(a_1^{n-2m})), c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m})) =\\ &A(y_1^m, A(b_1^m, a_1^{n-2m}, \mathsf{E}(a_1^{n-2m})), c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m})) \overset{(2\mathrm{R})}{\Longrightarrow}\\ &A(x_1^m, b_1^m, c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m})) =\\ &A(y_1^m, b_1^m, c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m})) \overset{(2\mathrm{R})}{\Longrightarrow} x_1^m = y_1^m.\end{aligned}$$

The proof of the statement 2°: By 1°, (1L) and by Prop. 2.3.
Sketch of the proof of 3°:

$$A(b_1^m, c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m}), \mathsf{E}(c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m}))) \overset{\widehat{(2\mathrm{L})}}{=} \mathsf{E}(c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m})),$$

$$A(b_1^m, c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m}), \mathsf{E}(c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m}))) \overset{(2\mathrm{R})}{=} b_1^m.$$

Sketch of the proof of $4°$*:*

$$\begin{aligned}
&A(b_1^m, x_1^m, a_1^{n-2m}) = A(b_1^m, y_1^m, a_1^{n-2m}) \Rightarrow\\
&A(c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m}), A(b_1^m, x_1^m, a_1^{n-2m}), \mathsf{E}(a_1^{n-2m})) =\\
&A(c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m}), A(b_1^m, y_1^m, a_1^{n-2m}), \mathsf{E}(a_1^{n-2m})) \overset{2°}{\Longrightarrow}\\
&A(c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m}), b_1^m, A(x_1^m, a_1^{n-2m}, \mathsf{E}(a_1^{n-2m}))) =\\
&A(c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m}), b_1^m, A(y_1^m, a_1^{n-2m}, \mathsf{E}(a_1^{n-2m}))) \overset{\widehat{(2R)}}{\Longrightarrow}\\
&A(c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m}), b_1^m, x_1^m) =\\
&A(c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m}), b_1^m, y_1^m) \overset{3°}{\Longrightarrow}\\
&A(c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m}), \mathsf{E}(c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m})), x_1^m) =\\
&A(c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m}), \mathsf{E}(c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m})), y_1^m) \overset{\widehat{(2L)}}{\Longrightarrow}\\
&x_1^m = y_1^m.
\end{aligned}$$

Sketch of the proof of $5°$*:*

$$\begin{aligned}
&A(a_1^{n-2m}, x_1^m, b_1^m) = A(a_1^{n-2m}, y_1^m, b_1^m) \Rightarrow\\
&A(c_1^{2m}, A(a_1^{n-2m}, x_1^m, b_1^m), d_1^{n-3m}) =\\
&A(c_1^{2m}, A(a_1^{n-2m}, y_1^m, b_1^m), d_1^{n-3m}) \overset{2°}{\Longrightarrow}\\
&A(A(c_1^{2m}, a_1^{n-2m}), x_1^m, b_1^m, d_1^{n-3m}) =\\
&A(A(c_1^{2m}, a_1^{n-2m}), y_1^m, b_1^m, d_1^{n-3m}) \overset{4°}{\Longrightarrow} x_1^m = y_1^m.
\end{aligned}$$

Sketch of the proof of $6°$*:*

$$\begin{aligned}
&A(b_1^m, x_1^m, a_1^{n-2m}) = d_1^m \overset{5°}{\Longleftrightarrow}\\
&A(c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m}), A(b_1^m, x_1^m, a_1^{n-2m}), \mathsf{E}(a_1^{n-2m})) =\\
&A(c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m}), d_1^m, \mathsf{E}(a_1^{n-2m})) \overset{2°}{\Longleftrightarrow}\\
&A(c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m}), b_1^m, A(x_1^m, a_1^{n-2m}, \mathsf{E}(a_1^{n-2m}))) =\\
&A(c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m}), d_1^m, \mathsf{E}(a_1^{n-2m})) \overset{\widehat{(2R)}}{\Longleftrightarrow}\\
&A(c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m}), b_1^m, x_1^m) =\\
&A(c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m}), d_1^m, \mathsf{E}(b_1^m, c_1^{n-3m})) \overset{3°}{\Longleftrightarrow}\\
&A(c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m}), \mathsf{E}(c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m})), x_1^m) =\\
&A(c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m}), d_1^m, \mathsf{E}(a_1^{n-2m})) \overset{\widehat{(2L)}}{\Longleftrightarrow}\\
&x_1^m = A(c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m}), d_1^m, \mathsf{E}(a_1^{n-2m})).
\end{aligned}$$

Finally, by 2°, 6° and by Prop. 2.4, we conclude that $(Q;A)$ is an (n,m)–group. $\square$

Similarly, one could prove also the following proposition:

Theorem 3.2. *Let $n \geq 3m$ and let $(Q;A)$ be an (n,m)–groupoid. Then, $(Q;A)$ is an (n,m)–group iff there is a mapping E of the set Q^{n-2m} into the set Q^m such that the laws*

$$A(x_1^{n-m-1}, A(x_{n-m}^{2n-m-1}), x_{2n-m}) = A(x_1^{n-m}, A(x_{n-m+1}^{2n-m})), \tag{1R}$$

$$A(a_1^{n-2m}, A(b_1^m, c_1^{n-m}), d_1^m) = A(a_1^{n-2m}, b_1^m, A(c_1^{n-m}, d_1^m)), \tag{1Rm}$$

$$A(\mathsf{E}(a_1^{n-2m}), a_1^{n-2m}, x_1^m) = x_1^m \tag{2L}$$

$$A(x_1^m, \mathsf{E}(a_1^{n-2m}), a_1^{n-2m}) = x_1^m. \tag{$\widehat{2\mathrm{R}}$}$$

Remark 3.2. For $m = 1$: (1R)=(1Rm).

Remark 3.3. The case $m = 1$ is described in [4]. See, also Chapter XII-1 in [9].

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